

# Hom-Lie 2-algebras \*

Danhua Chen

Department of Mathematics, Jilin University,

Yunhe Sheng

Department of Mathematics, Jilin University,

Changchun 130012, Jilin, China

email: shengyh@jlu.edu.cn

## Abstract

In this paper, we introduce the notions of hom-Lie 2-algebras, which is the categorification of hom-Lie algebras,  $HL_\infty$ -algebras, which is the hom-analogue of  $L_\infty$ -algebras, and crossed modules of hom-Lie algebras. We prove that the category of hom-Lie 2-algebras and the category of 2-term  $HL_\infty$ -algebras are equivalent. We give a detailed study on skeletal hom-Lie 2-algebras. In particular, we construct the hom-analogues of the string Lie 2-algebras associated to any semisimple involutive hom-Lie algebras. We also proved that there is a one-to-one correspondence between strict hom-Lie 2-algebras and crossed modules of hom-Lie algebras. We give the construction of strict hom-Lie 2-algebras from hom-left-symmetric algebras and symplectic hom-Lie algebras.

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<sup>0</sup> *Keyword:* hom-Lie algebras, quadratic hom-Lie algebras, hom-Lie 2-algebras,  $HL_\infty$ -algebras, crossed module of hom-Lie algebras, hom-left-symmetric algebras, symplectic hom-Lie algebras

<sup>0</sup> *MSC:* 17B99, 55U15.

\*Research partially supported by NSFC (11026046, 11101179), SRFDP (20100061120096) and “the Fundamental Research Funds for the Central Universities” (200903294).

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# 1 Introduction

The notion of hom-Lie algebras was introduced by Hartwig, Larsson, and Silvestrov in [5] as part of a study of deformations of the Witt and the Virasoro algebras. In a hom-Lie algebra, the Jacobi identity is twisted by a linear map, called the hom-Jacobi identity. Some  $q$ -deformations of the Witt and the Virasoro algebras have the structure of a hom-Lie algebra [5]. Because of close relation to discrete and deformed vector fields and differential calculus [5, 6, 7], hom-Lie algebras are widely studied recently [4, 8, 9, 11, 14, 15, 16, 17].

Recently, people have payed more attention to higher categorical structures with motivations from string theory [3]. One way to provide higher categorical structures is by categorifying existing mathematical concepts. One of the simplest higher structure is a 2-vector space, which is a categorified vector space. If we further put Lie algebra structures on 2-vector spaces, then we obtain the notion of Lie 2-algebras [2]. The Jacobi identity is replaced by a natural transformation, called Jacobiator, which also satisfies some coherence laws of its own. One of the motivating examples is the differentiation of Witten's string Lie 2-group  $String(n)$ , which is called a string Lie 2-algebra. As  $SO(n)$  is the connected part of  $O(n)$  and  $Spin(n)$  is the simply connected cover of  $SO(n)$ ,  $String(n)$  is a "cover" of  $Spin(n)$  which has trivial  $\pi_3$  (notice that  $\pi_2(G) = 0$  for any Lie group  $G$ ). The differentiation of  $String(n)$  is not any more  $\mathfrak{so}(n)$ , but a central extension of  $\mathfrak{so}(n)$  by the abelian Lie 2-algebra  $\mathbb{R} \rightarrow 0$ , which is a Lie 2-algebra by itself. The concept of string Lie 2-algebra is later generalized to any such extension of a semisimple Lie algebra.  $L_\infty$ -algebras, sometimes called strongly homotopy (sh) Lie algebras, were introduced [10] as a model for "Lie algebras that satisfy Jacobi identity up to all higher homotopies". It is well known that Lie 2-algebras are equivalent to 2-term  $L_\infty$ -algebras.

In this paper, we provide the categorification of hom-Lie algebras, which we call hom-Lie 2-algebras. We also give the hom-analogue of  $L_\infty$ -algebras, which we call  $HL_\infty$ -algebras. The main difficulty to give these definitions is how to let the hom-structures involved in. In the case of Lie 2-algebras (or 2-term  $L_\infty$ -algebras), the Jacobiator (or  $l_3$ ) should satisfy some kind of closed condition. Motivated by the cohomology theory introduced in [11], we solve this difficulty successfully. We prove that the category of hom-Lie 2-algebras and the category of 2-term  $HL_\infty$ -algebras are equivalent. Skeletal hom-Lie 2-algebras are studied in detail. We give their classification by the third cohomology of hom-Lie algebras, and provide examples from quadratic hom-Lie algebras introduced in [4] by Benayadi and Makhlouf. In particular, we introduce the hom-analogues of the string Lie 2-algebras. The notion of crossed modules of hom-Lie algebras is also introduced and we prove that there is a one-to-one correspondence between strict hom-Lie 2-algebras and crossed modules of hom-Lie algebras. We construct strict hom-Lie 2-algebras from hom-left-symmetric algebras. Furthermore, we introduce the notion of a symplectic hom-Lie algebra, which is a hom-Lie algebra together with a symplectic form, and give the construction of strict hom-Lie 2-algebras from symplectic hom-Lie algebras.

The paper is organized as follows. In Section 2, we recall some necessary background knowledge, including the cohomology theory of hom-Lie algebras and 2-vector spaces. In Section 3, first we give the definition of hom-Lie 2-algebras, which is the categorification of hom-Lie algebras. Then we introduce the hom-analogue of  $L_\infty$ -algebras, which we call  $HL_\infty$ -algebras. We give

the definition of 2-term  $HL_\infty$ -algebras by explicit formulas. At last, we prove that the category of hom-Lie 2-algebras and the category of 2-term  $HL_\infty$ -algebras are equivalent. In Section 4, we study skeletal hom-Lie 2-algebras. Especially, we construct examples of skeletal hom-Lie 2-algebras from involutive quadratic hom-Lie algebras, and obtain hom-analogues of string Lie 2-algebras. In Section 5, first we introduce the notion of crossed modules of hom-Lie algebras, and prove that they are equivalent to strict hom-Lie 2-algebras. We construct strict hom-Lie 2-algebras from hom-left-symmetric algebras. At last, we introduce the notion of symplectic hom-Lie algebras. There is a natural hom-left-symmetric algebra associated to a symplectic hom-Lie algebra, such that it is the sub-adjacent hom-Lie algebra of the induced hom-left-symmetric algebra. Then we give the construction of strict hom-Lie 2-algebras from symplectic hom-Lie algebras.

## 2 Preliminaries

In this section, we recall some basic notions and facts about hom-Lie algebras [5] and 2-vector spaces [2].

### • Hom-Lie algebras and their representations

**Definition 2.1.** [5] A hom-Lie algebra<sup>1</sup> is a triple  $(\mathfrak{g}, [\cdot, \cdot]_\mathfrak{g}, \phi_\mathfrak{g})$  consisting of a linear space  $\mathfrak{g}$ , a skew-symmetric bilinear map (bracket)  $[\cdot, \cdot]_\mathfrak{g} : \wedge^2 \mathfrak{g} \longrightarrow \mathfrak{g}$  and an algebra morphism  $\phi_\mathfrak{g} : \mathfrak{g} \longrightarrow \mathfrak{g}$  satisfying

$$[\phi_\mathfrak{g}(u), [v, w]_\mathfrak{g}]_\mathfrak{g} + [\phi_\mathfrak{g}(v), [w, u]_\mathfrak{g}]_\mathfrak{g} + [\phi_\mathfrak{g}(w), [u, v]_\mathfrak{g}]_\mathfrak{g} = 0. \quad (1)$$

The hom-Lie algebra  $(\mathfrak{g}, [\cdot, \cdot]_\mathfrak{g}, \phi_\mathfrak{g})$  is said to be regular (involutive), if  $\phi_\mathfrak{g}$  is nondegenerate (satisfies  $\phi_\mathfrak{g}^2 = \text{Id}$ );

**Definition 2.2.** A morphism of hom-Lie algebras  $f : (\mathfrak{g}, [\cdot, \cdot]_\mathfrak{g}, \phi_\mathfrak{g}) \longrightarrow (\mathfrak{k}, [\cdot, \cdot]_\mathfrak{k}, \phi_\mathfrak{k})$  is a linear map  $f : \mathfrak{g} \longrightarrow \mathfrak{k}$  such that

$$f[u, v]_\mathfrak{g} = [f(u), f(v)]_\mathfrak{k}, \quad \forall u, v \in \mathfrak{g}, \quad (2)$$

$$f \circ \phi_\mathfrak{g} = \phi_\mathfrak{k} \circ f. \quad (3)$$

Let  $(\mathfrak{g}, [\cdot, \cdot]_\mathfrak{g}, \phi_\mathfrak{g})$  be a hom-Lie algebra and  $V$  an arbitrary vector space. Let  $A \in \mathfrak{gl}(V)$  be an arbitrary linear transformation from  $V$  to  $V$ . The representation of hom-Lie algebras was introduced in [11].

**Definition 2.3.** A representation of the hom-Lie algebra  $(\mathfrak{g}, [\cdot, \cdot]_\mathfrak{g}, \phi_\mathfrak{g})$  on the vector space  $V$  with respect to  $A \in \mathfrak{gl}(V)$  is a linear map  $\rho_A : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ , such that for any  $u, v \in \mathfrak{g}$ , the following equalities are satisfied:

$$(i) \quad \rho_A(\phi_\mathfrak{g}(u)) \circ A = A \circ \rho_A(u);$$

$$(ii) \quad \rho_A([u, v]_\mathfrak{g}) \circ A = \rho_A(\phi_\mathfrak{g}(u)) \circ \rho_A(v) - \rho_A(\phi_\mathfrak{g}(v)) \circ \rho_A(u).$$

The set of  $k$ -cochains on  $\mathfrak{g}$  with values in  $V$ , which we denote by  $C^k(\mathfrak{g}; V)$ , is the set of skew-symmetric  $k$ -linear maps from  $\mathfrak{g} \times \cdots \times \mathfrak{g}$  ( $k$ -times) to  $V$ :

$$C^k(\mathfrak{g}; V) \triangleq \{f : \wedge^k \mathfrak{g} \longrightarrow V \text{ is a } k\text{-linear map}\}.$$

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<sup>1</sup>The hom-Lie algebras defined here are also called multiplicative hom-Lie algebras in some references

A  $k$ -hom-cochain on  $\mathfrak{g}$  with values in  $V$  is defined to be a  $k$ -cochain  $f \in C^k(\mathfrak{g}; V)$  such that it is compatible with  $\phi_{\mathfrak{g}}$  and  $A$  in the sense that  $A \circ f = f \circ \phi_{\mathfrak{g}}$ , i.e.

$$A(f(u_1, \dots, u_k)) = f(\phi_{\mathfrak{g}}(u_1), \dots, \phi_{\mathfrak{g}}(u_k)).$$

Denote by  $C_{\phi_{\mathfrak{g}}, A}^k(\mathfrak{g}; V)$  the set of  $k$ -hom-cochains:

$$C_{\phi_{\mathfrak{g}}, A}^k(\mathfrak{g}; V) \triangleq \{f \in C^k(\mathfrak{g}; V) \mid A \circ f = f \circ \phi_{\mathfrak{g}}\}.$$

In [11], the author defined the coboundary operator  $d_{\rho_A} : C_{\phi_{\mathfrak{g}}, A}^k(\mathfrak{g}; V) \longrightarrow C_{\phi_{\mathfrak{g}}, A}^{k+1}(\mathfrak{g}; V)$  by setting

$$\begin{aligned} d_{\rho_A} f(u_1, \dots, u_{k+1}) &= \sum_{i=1}^{k+1} (-1)^{i+1} \rho(\phi_{\mathfrak{g}}^{k-1}(u_i))(f(u_1, \dots, \widehat{u}_i, \dots, u_{k+1})) \\ &\quad + \sum_{i < j} (-1)^{i+j} f([u_i, u_j]_{\mathfrak{g}}, \phi_{\mathfrak{g}}(u_1) \cdots, \widehat{u}_i, \dots, \widehat{u}_j, \dots, \phi_{\mathfrak{g}}(u_{k+1})). \end{aligned} \quad (4)$$

The equality  $d_{\rho_A}^2 = 0$  was proved in [11]. Thus, we can obtain the cohomology of hom-Lie algebras.

**Remark 2.4.** *The formula given by (4) is slightly different from the formula given in [11], where the author use  $\rho(\phi_{\mathfrak{g}}^k(u_i))$ . Both of them are correct. All the results in [11] hold after a small modification for the coboundary operator  $d_{\rho_A}$  given by (4).*

Every hom-Lie algebra has the trivial representation on  $\mathbb{R}$  with respect to  $\text{Id} : \mathbb{R} \longrightarrow \mathbb{R}$ , the corresponding coboundary operator, which we denote by  $d_T$ , is given by

$$d_T f(u_1, \dots, u_{k+1}) = \sum_{i < j} (-1)^{i+j} f([u_i, u_j]_{\mathfrak{g}}, \phi_{\mathfrak{g}}(u_1) \cdots, \widehat{u}_i, \dots, \widehat{u}_j, \dots, \phi_{\mathfrak{g}}(u_{k+1})).$$

Denote by  $Z_{\phi_{\mathfrak{g}}}^k(\mathfrak{g})$  and  $B_{\phi_{\mathfrak{g}}}^k(\mathfrak{g})$  the corresponding closed  $k$ -hom-cochains and exact  $k$ -hom-cochains respectively. Denote the resulting cohomology by  $H^k(\mathfrak{g})$ .

## • 2-vector spaces

Vector spaces can be categorified to 2-vector spaces. A good introduction for this subject is [2]. Let  $\text{Vect}$  be the category of vector spaces.

**Definition 2.5.** [2] *A 2-vector space is a category in the category  $\text{Vect}$ .*

Thus, a 2-vector space  $C$  is a category with a vector space of objects  $C_0$  and a vector space of morphisms  $C_1$ , such that all the structure maps are linear. Let  $i : C_0 \longrightarrow C_1$  be the identity assigning map and  $s, t : C_1 \longrightarrow C_0$  be the source and target maps respectively. Let  $\cdot_{\vee}$  be the composition of morphisms.

It is well known that the 2-category of 2-vector spaces is equivalent to the 2-category of 2-term complexes of vector spaces. Roughly speaking, given a 2-vector space  $C$ ,

$$\text{Ker}(s) \xrightarrow{t} C_0 \quad (5)$$

is a 2-term complex. Conversely, any 2-term complex of vector spaces  $V_1 \xrightarrow{d} V_0$  gives rise to a 2-vector space of which the set of objects is  $V_0$ , the set of morphisms is  $V_0 \oplus V_1$ , the source map  $s$

is given by  $s(v, m) = v$ , and the target map  $t$  is given by  $t(v, m) = v + dm$ , where  $v \in V_0$ ,  $m \in V_1$ . We denote the 2-vector space associated to the 2-term complex of vector spaces  $V_1 \xrightarrow{d} V_0$  by  $\mathbb{V}$ :

$$\mathbb{V} = \begin{array}{c} \mathbb{V}_1 := V_0 \oplus V_1 \\ s \downarrow \quad \downarrow t \\ \mathbb{V}_0 := V_0. \end{array} \quad (6)$$

Given a 2-vector space  $\mathbb{V}$ , we define  $\text{End}_d^0(\mathbb{V})$  by

$$\text{End}_d^0(\mathbb{V}) \triangleq \{(A_0, A_1) \in \mathfrak{gl}(V_0) \oplus \mathfrak{gl}(V_1) | A_0 \circ d = d \circ A_1\},$$

and define  $\text{End}^1(\mathbb{V}) \triangleq \text{Hom}(V_0, V_1)$ . Then we have,

**Lemma 2.6.** [12]  $\text{End}_d^0(\mathbb{V})$  is the space of linear functors from  $\mathbb{V}$  to  $\mathbb{V}$ .

There is a differential  $\delta : \text{End}^1(\mathbb{V}) \longrightarrow \text{End}_d^0(\mathbb{V})$  given by

$$\delta(\alpha) \triangleq (d \circ \alpha, \alpha \circ d), \quad \forall \alpha \in \text{End}^1(\mathbb{V}),$$

and a bracket operation  $[\cdot, \cdot]_C$  given by the graded commutator. More precisely, for any  $A = (A_0, A_1), B = (B_0, B_1) \in \text{End}_d^0(\mathbb{V})$  and  $\alpha \in \text{End}^1(\mathbb{V})$ ,  $[\cdot, \cdot]_C$  is given by

$$[A, B]_C = A \circ B - B \circ A = (A_0 \circ B_0 - B_0 \circ A_0, A_1 \circ B_1 - B_1 \circ A_1),$$

and

$$[A, \alpha]_C = A \circ \alpha - \alpha \circ A = (A_1 \circ \alpha - \alpha \circ A_0, \quad (7)$$

These two operations make  $\text{End}^1(\mathbb{V}) \xrightarrow{\delta} \text{End}_d^0(\mathbb{V})$  into a 2-term DGLA (proved in [13]), which we denote by  $\text{End}(\mathbb{V})$ . It plays the same role as  $\mathfrak{gl}(V)$  for a vector space  $V$  in the classical case.

### 3 Hom-Lie 2-algebras and $HL_\infty$ -algebras

In this section, first we category the notion of hom-Lie algebras, and obtain the hom-Lie 2-algebras. Then we give the hom-analogue of  $L_\infty$ -algebras, what we call  $HL_\infty$ -algebras. We give the structure of a 2-term  $HL_\infty$ -algebra by explicit formulas. At last, we prove that the category of hom-Lie 2-algebras and the category of 2-term  $HL_\infty$ -algebras are equivalent.

#### 3.1 Hom-Lie 2-algebras

**Definition 3.1.** A hom-Lie 2-algebra is a 2-vector space  $L$  equipped with

- a skew-symmetric bilinear functor, the bracket,  $[\cdot, \cdot] : L \times L \longrightarrow L$ ,
- a linear functor  $\Phi = (\Phi_0, \Phi_1) : L \longrightarrow L$  satisfying:

$$\Phi[\xi, \eta] = [\Phi(\xi), \Phi(\eta)], \quad \forall \xi, \eta \in L. \quad (8)$$

- a skew-symmetric trilinear natural isomorphism, the hom-Jacobiator,

$$J_{x,y,z} : [[x, y], \Phi_0(z)] \longrightarrow [\Phi_0(x), [y, z]] + [[x, z], \Phi_0(y)],$$

satisfying  $J_{\Phi_0(x), \Phi_0(y), \Phi_0(z)} = \Phi_1 J_{x,y,z}$ ,

such that the following hom-Jacobiator identity is satisfied,

$$\begin{aligned}
& J_{[w,x],\Phi_0(y),\Phi_0(z)} \cdot_v ([J_{w,x,z}, \Phi_0^2(y)] + 1) \cdot_v (J_{\Phi_0(w),[x,z],\Phi_0(y)} + J_{[w,z],\Phi_0(x),\Phi_0(y)} + 1) \\
= & [J_{w,x,y}, \Phi_0^2(z)] \cdot_v (J_{[w,y],\Phi_0(x),\Phi_0(z)} + J_{\Phi_0(w),[x,y],\Phi_0(z)}) \\
& \cdot_v ([\Phi_0^2(w), J_{x,y,z}] + [J_{w,y,z}, \Phi_0^2(x)] + 1) \cdot_v (1 + J_{\Phi_0(w),[y,z],\Phi_0(x)}),
\end{aligned}$$

or, in terms of a diagram,

$$\begin{array}{ccc}
[[[w,x],\Phi_0(y)],\Phi_0^2(z)] & \xrightarrow{J_{[w,x],\Phi_0(y),\Phi_0(z)}} & [\Phi_0[w,x],[\Phi_0(y),\Phi_0(z)]] + [[[w,x],\Phi_0(z)],\Phi_0^2(y)] \\
\downarrow [J_{w,x,y},\Phi_0^2(z)] & & \downarrow 1 + [J_{w,x,z},\Phi_0^2(y)] \\
[[\Phi_0(w),[x,y]],\Phi_0^2(z)] + [[[w,y],\Phi_0(x)],\Phi_0^2(z)] & & M \\
\downarrow J_{\Phi_0(w),[x,y],\Phi_0(z)} + J_{[w,y],\Phi_0(x),\Phi_0(z)} & & \downarrow 1 + J_{\Phi_0(w),[x,z],\Phi_0(y)} + J_{[w,z],\Phi_0(x),\Phi_0(y)} \\
P & \xrightarrow{(1 + [\Phi_0^2(w),J_{x,y,z}] + [J_{w,y,z},\Phi_0^2(x)]) \cdot_v (1 + J_{\Phi_0(w),[y,z],\Phi_0(x)})} & Q,
\end{array}$$

where  $M, P$  and  $Q$  are given by

$$\begin{aligned}
M &= [\Phi_0[w,x], [\Phi_0(y), \Phi_0(z)]] + [[\Phi_0(w), [x,z]], \Phi_0^2(y)] + [[[w,z], \Phi_0(x)], \Phi_0^2(y)]; \\
P &= [\Phi_0^2(w), [[x,y], \Phi_0(z)]] + [[\Phi_0(w), \Phi_0(z)], \Phi_0[x,y]] \\
&\quad + [\Phi_0[w,y], [\Phi_0(x), \Phi_0(z)]] + [[[w,y], \Phi_0(z)], \Phi_0^2(x)]; \\
Q &= [\Phi_0[w,x], [\Phi_0(y), \Phi_0(z)]] + [\Phi_0^2(w), [[x,z], \Phi_0(y)]] + [[\Phi_0(w), \Phi_0(y)], \Phi_0([x,z])] \\
&\quad + [\Phi_0([w,z]), [\Phi_0(x), \Phi_0(y)]] + [[[w,z], \Phi_0(y)], \Phi_0^2(x)].
\end{aligned}$$

Usually we denote a hom-Lie 2-algebra by  $(L, [\cdot, \cdot], J, \Phi)$ . A hom-Lie 2-algebra is called *strict* if the hom-Jacobiator is the identity isomorphism.

**Definition 3.2.** Given hom-Lie 2-algebras  $(L, [\cdot, \cdot], \Phi)$  and  $(L', [\cdot, \cdot]', \Phi')$ , a hom-Lie 2-algebra morphism  $F : L \longrightarrow L'$  consists of:

- a linear functor  $(F_0, F_1)$  from the underlying 2-vector space of  $L$  to that of  $L'$  such that

$$\Phi' \circ (F_0, F_1) = (F_0, F_1) \circ \Phi,$$

- a skew-symmetric bilinear natural transformation

$$F_2(x, y) : [F_0(x), F_0(y)]' \longrightarrow F_0([x, y])$$

satisfying  $F_2(\Phi_0(x), \Phi_0(y)) = \Phi'_1 F_2(x, y)$ , such that the following diagram commutes:

$$\begin{array}{ccc}
[[F_0(x), F_0(y)]', \Phi'(F_0(z))]' & \xrightarrow{[F_2(x, y), 1]'} & [F_0[x, y], F_0\Phi(z)]' \\
\downarrow J_{F_0(x), F_0(y), F_0(z)} & & \downarrow F_2([x, y], \Phi(z)) \\
[\Phi'(F_0(x)), [F_0(y), F_0(z)]']' + [[F_0(x), F_0(z)]', \Phi'(F_0(y))]' & & F_0[[x, y], \Phi(z)] \\
\downarrow [1, F_2(y, z)]' + [F_2(x, z), 1]' & & \downarrow F_1 J_{x, y, z} \\
[F_0(\Phi(x)), F_0[y, z]]' + [F_0[x, z], F_0(\Phi(y))]' & \xrightarrow{F_2(\Phi(x), [y, z]) + F_2([x, z], \Phi(y))} & F_0[\Phi(x), [y, z]] + F_0[[x, z], \Phi(y)].
\end{array}$$

The identity morphism  $\text{Id}_L : L \longrightarrow L$  has the identity functor as its underlying functor, together with an identity natural transformation as  $(\text{Id}_L)_2$ . Let  $L$ ,  $L'$  and  $L''$  be hom-Lie 2-algebras, the composition of a pair of hom-Lie 2-algebra morphisms  $F : L \longrightarrow L'$  and  $G : L' \longrightarrow L''$ , which we denote by  $G \circ F$ , is given by letting the functor  $((G \circ F)_0, (G \circ F)_1)$  be the usual composition of  $(G_0, G_1)$  and  $(F_0, F_1)$ , and letting  $(G \circ F)_2$  be the following composite:

$$\begin{array}{ccc} [G_0 \circ F_0(x), G_0 \circ F_0(y)]'' & & \\ \downarrow G_2(F_0(x), F_0(y)) & \searrow (G \circ F)_2(x, y) & \\ & G_0 \circ F_0[x, y] & \\ & \nearrow G_1(F_0(x, y)) & \\ G_0[F_0(x), F_0(y)]' & & \end{array},$$

It is straightforward to see that

**Proposition 3.3.** *There is a category  $\mathbf{HLie2}$  with hom-Lie 2-algebras as objects and hom-Lie 2-algebra morphisms as morphisms.*

### 3.2 $HL_\infty$ -algebras

**Definition 3.4.** *A  $HL_\infty$ -algebra is a graded vector space  $V_\bullet = \bigoplus_{i=0}^\infty V_i$  equipped with*

- *a system  $\{l_k | 1 \leq k < \infty\}$  of linear maps  $l_k : \wedge^k V_\bullet \longrightarrow V_\bullet$  with  $\deg(l_k) = k - 2$ , where the exterior powers are interpreted in the graded sense, i.e. the following relation with Koszul sign “Ksgn” is satisfied:*

$$l_k(x_{\sigma(1)}, \dots, x_{\sigma(k)}) = \text{sgn}(\sigma) \text{Ksgn}(\sigma) l_k(x_1, \dots, x_k), \quad \forall \sigma \in S_k;$$

- *a system  $\{\phi_k | 1 \leq k < \infty\}$  of linear maps  $\phi_k : V_k \rightarrow V_k$ , such that for any  $x_1 \in V_{s_1}, \dots, x_k \in V_{s_k}$ , we have*

$$\phi_{(\sum s_i) + k - 2}(l_k(x_1, \dots, x_k)) = l_k(\phi_{s_1}(x_1), \dots, \phi_{s_k}(x_k)),$$

*such that the following generalized form of the Jacobi identity holds for all  $0 \leq n < \infty$ ,*

$$\sum_{i+j=n+1} \sum_{\sigma} (-1)^{i(j-1)} \text{sgn}(\sigma) \text{Ksgn}(\sigma) l_j(l_i(x_{\sigma(1)}, \dots, x_{\sigma(i)}), \phi_{m_{i+1}}^{i-1}(x_{\sigma(i+1)}), \dots, \phi_{m_n}^{i-1}(x_{\sigma(n)})) = 0,$$

*where  $x_{\sigma(i+1)} \in V_{m_{i+1}}, \dots, x_{\sigma(n)} \in V_{m_n}$ , and the summation is taken over all  $(i, n-i)$ -unshuffles with  $i \geq 1$ .*

For  $n = 1$ , we have

$$l_1^2 = 0, \quad l_1 : V_{i+1} \longrightarrow V_i,$$

which means that  $V_\bullet$  is a complex of vector spaces, so we write  $d = l_1$  as usual. For  $n = 2$ , we have

$$dl_2(x_p, x_q) = l_2(dx_p, x_q) + (-1)^p l_2(x_p, dx_q), \quad \forall x_p \in V_p, x_q \in V_q,$$

which means that  $d$  is a derivation with respect to  $l_2$ .

Constraint on the 2-term case, it is not hard to obtain:

**Proposition-Definition 3.5.** A 2-term  $HL_\infty$ -algebra  $\mathcal{V}$  consists of the following data:

- a complex of vector spaces  $V_1 \xrightarrow{d} V_0$ ,
- bilinear maps  $l_2, : V_i \times V_j \longrightarrow V_{i+j}$ ,
- two linear transformations  $\phi_0 \in \mathfrak{gl}(V_0)$  and  $\phi_1 \in \mathfrak{gl}(V_1)$  satisfying  $\phi_0 \circ d = d \circ \phi_1$ ,
- a skew-symmetric trilinear map  $l_3 : V_0 \times V_0 \times V_0 \longrightarrow V_1$  satisfying  $l_3 \circ \phi_0 = \phi_1 \circ l_3$ ,

such that for any  $w, x, y, z \in V_0$  and  $m, n \in V_1$ , the following equalities are satisfied:

- (a)  $l_2(x, y) = -l_2(y, x)$ ,
- (b)  $l_2(x, m) = -l_2(m, x)$ ,
- (c)  $l_2(m, n) = 0$ ,
- (d)  $dl_2(x, m) = l_2(x, dm)$ ,
- (e)  $l_2(dm, n) = l_2(m, dn)$ ,
- (f)  $\phi_0(l_2(x, y)) = l_2(\phi_0(x), \phi_0(y))$ ,
- (g)  $\phi_1(l_2(x, m)) = l_2(\phi_0(x), \phi_1(m))$ ,
- (h)  $dl_3(x, y, z) = l_2(\phi_0(x), l_2(y, z)) + l_2(\phi_0(y), l_2(z, x)) + l_2(\phi_0(z), l_2(x, y))$ ,
- (i)  $dl_3(x, y, dm) = l_2(\phi_0(x), l_2(y, m)) + l_2(\phi_0(y), l_2(m, x)) + l_2(\phi_1(m), l_2(x, y))$ ,
- (j)

$$\begin{aligned}
& l_3(l_2(w, x), \phi_0(y), \phi_0(z)) + l_2(l_3(w, x, z), \phi_0^2(y)) \\
& + l_3(\phi_0(w), l_2(x, z), \phi_0(y)) + l_3(l_2(w, z), \phi_0(x), \phi_0(y)) \\
= & l_2(l_3(w, x, y), \phi_0^2(z)) + l_3(l_2(w, y), \phi_0(x), \phi_0(z)) + l_3(\phi_0(w), l_2(x, y), \phi_0(z)) \\
& + l_2(\phi_0^2(w), l_3(x, y, z)) + l_2(l_3(w, y, z), \phi_0^2(x)) + l_3(\phi_0(w), l_2(y, z), \phi_0(x)).
\end{aligned}$$

We will denote a 2-term  $HL_\infty$ -algebra by  $(V_1 \xrightarrow{d} V_0, l_2, l_3, \phi_0, \phi_1)$ .

**Definition 3.6.** Let  $\mathcal{V}$  and  $\mathcal{V}'$  be 2-term  $HL_\infty$ -algebras. A  $HL_\infty$ -morphism  $f : \mathcal{V} \longrightarrow \mathcal{V}'$  consists of:

- a chain map  $f : \mathcal{V} \longrightarrow \mathcal{V}'$ , which consists of linear maps  $f_0 : V_0 \longrightarrow V'_0$  and  $f_1 : V_1 \longrightarrow V'_1$  satisfying

$$f_0 \circ d = d' \circ f_1,$$

and

$$f_0 \circ \phi_0 = \phi'_0 \circ f_0, \quad f_1 \circ \phi_1 = \phi'_1 \circ f_1. \quad (9)$$

- a skew-symmetric bilinear map  $f_2 : V_0 \times V_0 \longrightarrow V'_1$  satisfying  $f_2(\phi_0(x), \phi_0(y)) = \phi'_1 f_2(x, y)$ ,

such that for all  $x, y, z \in V_0$  and  $m, n \in V_1$ , we have

- $df_2(x, y) = f_0(l_2(x, y)) - l'_2(f_0(x), f_0(y))$ ,



- $f_2(x, dm) = f_1(l_2(x, m)) - l'_2(f_0(x), f_1(m)),$

- 

$$\begin{aligned}
& l'_2(f_2(x, y), f_0(\phi_0(z))) + f_2(l_2(x, y), \phi_0(z)) + f_1(l_3(x, y, z)) \\
= & l'_3(f_0(x), f_0(y), f_0(z)) + l'_2(f_0(\phi_0(x)), f_2(y, z)) + l'_2(f_2(x, z), f_0(\phi_0(y))) \\
& + f_2(\phi_0(x), l_2(y, z)) + f_2(l_2(x, z), \phi_0(y)).
\end{aligned} \tag{10}$$

The identity  $HL_\infty$ -morphism  $\text{Id}_\mathcal{V} : \mathcal{V} \longrightarrow \mathcal{V}$  has the identity chain map as its underlying map, together with  $(\text{Id}_\mathcal{V})_2 = 0$ , i.e.  $\text{Id}_\mathcal{V} = (\text{Id}_{V_0}, \text{Id}_{V_1}, 0)$ . Let  $\mathcal{V}$ ,  $\mathcal{V}'$  and  $\mathcal{V}''$  be  $HL_\infty$ -algebras, and  $f : \mathcal{V} \rightarrow \mathcal{V}'$  and  $f' : \mathcal{V}' \rightarrow \mathcal{V}''$  be  $HL_\infty$ -morphisms, we define their composition  $f' \circ f = ((f' \circ f)_0, (f' \circ f)_1, (f' \circ f)_2)$  by setting  $(f' \circ f)_0 = f'_0 \circ f_0$ ,  $(f' \circ f)_1 = f'_1 \circ f_1$ , and

$$(f' \circ f)_2(x, y) = f'_2(f_0(x), f_0(y)) + f'_1(f_2(x, y)).$$

This is exactly the same as the composition of  $L_\infty$ -morphisms between 2-term  $L_\infty$ -algebras. To see that it is indeed a  $HL_\infty$ -morphism, we still need to show that the conditions related with  $\phi_0$  and  $\phi_1$  in Definition 3.6 hold. We leave it as an exercise.

Thus, we have

**Proposition 3.7.** *There is a category  $\mathbf{2HL}_\infty$  with 2-term  $HL_\infty$ -algebras as objects and  $HL_\infty$ -morphisms as morphisms.*

### 3.3 The equivalence of hom-Lie 2-algebras and 2-term $HL_\infty$ -algebras

**Theorem 3.8.** *The categories  $\mathbf{2HL}_\infty$  and  $\mathbf{HLie2}$  are equivalent.*

**Proof.** We only give a sketch of the proof. First we construct a functor  $T : \mathbf{2HL}_\infty \longrightarrow \mathbf{HLie2}$ . Given a 2-term  $HL_\infty$ -algebra  $\mathcal{V} = (V_1 \xrightarrow{d} V_0, l_2, l_3, \phi_0, \phi_1)$ , we have a 2-vector space  $L$  given by (6). Define the skew-symmetric bilinear functor  $[\cdot, \cdot] : L \times L \longrightarrow L$  by

$$[(x, m), (y, n)] = (l_2(x, y), l_2(x, n) + l_2(m, y) + l_2(dm, n)), \quad \forall (x, m), (y, n) \in L_1 = V_0 \oplus V_1.$$

Define the linear functor  $\Phi$  by

$$\Phi = (\Phi_0, \Phi_1) = (\phi_0, \phi_0 \oplus \phi_1).$$

By the fact that  $\phi_0$  and  $\phi_1$  commutes with the differential  $d$ , we deduce that  $\Phi$  is a functor, i.e.  $\Phi \in \text{End}(L)$ . By Condition (f) and (g) in Definition 3.5, we have

$$\begin{aligned}
\Phi[(x, m), (y, n)] &= (\phi_0 l_2(x, y), \phi_1(l_2(x, n) + l_2(m, y) + l_2(dm, n))) \\
&= (l_2(\phi_0(x), \phi_0(y)), l_2(\phi_0(x), \phi_1(n)) + l_2(\phi_1(m), \phi_0(y)) + l_2(\phi_0(dm), \phi_1(n))) \\
&= (l_2(\phi_0(x), \phi_0(y)), l_2(\phi_0(x), \phi_1(n)) + l_2(\phi_1(m), \phi_0(y)) + l_2(d\phi_1(m), \phi_1(n))) \\
&= [(\phi_0(x), \phi_1(m)), (\phi_0(y), \phi_1(n))] \\
&= [\Phi(x, m), \Phi(y, n)].
\end{aligned}$$

Define the Jacobiator by

$$J_{x, y, z} = ([x, y], \phi_0(z), l_3(x, y, z)).$$

It is straightforward to deduce that

$$\begin{aligned}
J_{\Phi_0(x), \Phi_0(y), \Phi_0(z)} &= ([[\phi_0(x), \phi_0(y)], \phi_0^2(z)], l_3(\phi_0(x), \phi_0(y), \phi_0(z))) \\
&= (\phi_0[[x, y], \phi_0(z)], \phi_1 l_3(x, y, z)) \\
&= \Phi_1 J_{x, y, z}.
\end{aligned}$$

By the various conditions of  $(V_1 \xrightarrow{d} V_0, l_2, l_3, \phi_0, \phi_1)$  being a 2-term  $HL_\infty$ -algebra, we deduce that  $(L, [\cdot, \cdot], J, \Phi)$  is a hom-Lie 2-algebra. Thus, we have constructed a hom-Lie 2-algebra  $L = T(\mathcal{V})$  from a 2-term  $HL_\infty$ -algebra  $\mathcal{V}$ .

For any  $HL_\infty$ -morphism  $f = (f_0, f_1, f_2)$  from  $\mathcal{V}$  to  $\mathcal{V}'$ , next we construct a hom-Lie 2-algebra morphism  $F = T(f)$  from  $L = T(\mathcal{V})$  to  $L' = T(\mathcal{V}')$ .

Let  $F_0 = f_0$ ,  $F_1 = f_0 \oplus f_1$ , and  $F_2$  be given by

$$F_2(x, y) = ([f_0(x), f_0(y)], f_2(x, y)).$$

Then  $F_2(x, y)$  is a bilinear skew-symmetric natural isomorphism from  $[F_0(x), F_0(y)]$  to  $F_0[x, y]$ , and  $F = (F_0, F_1, F_2)$  is a morphism from  $L$  to  $L'$ .

One can also deduce that  $T$  preserves the identity morphisms and the composition of morphisms. Thus,  $T$  constructed above is a functor from **2HL** $_\infty$  to **HLie2**.

Conversely, given a hom-Lie 2-algebra  $L$ , we construct the 2-term  $HL_\infty$ -algebra  $\mathcal{V} = S(L)$  as follows. As a complex of vector spaces,  $\mathcal{V}$  is obtained by (5), i.e.  $V_0 = L_0$ ,  $V_1 = \text{Ker}(s)$ , and  $d = t|_{\text{Ker}(s)}$ . Define  $l_2$  by

$$l_2(x, y) = [x, y], \quad l_2(x, m) = -l_2(m, x) = [i(x), m], \quad l_2(m, n) = 0,$$

Define  $\phi_0 = \Phi_0 : V_0 (= L_0) \longrightarrow V_0$ , and define  $\phi_1 = \Phi_1|_{V_1 = \text{Ker}(s)} : V_1 \longrightarrow V_1$ . Since  $\Phi$  is a functor, we have  $\phi_0 \circ d = d \circ \phi_1$ . Since  $\Phi$  satisfies (8), it follows that  $\phi_0$  and  $\phi_1$  satisfy Conditions (f) and (g) in Definition 3.5.

Furthermore, define  $l_3$  by

$$l_3(x, y, z) = J_{x, y, z} - i(s(J_{x, y, z})).$$

Since  $J_{\Phi_0(x), \Phi_0(y), \Phi_0(z)} = \Phi_1 J_{x, y, z}$ , we deduce that  $\phi_1 l_3(x, y, z) = l_3(\phi_0(x), \phi_0(y), \phi_0(z))$ . The various conditions of  $L$  being a hom-Lie 2-algebra imply that  $\mathcal{V}$  is 2-term  $HL_\infty$ -algebra.

Let  $F = (F_0, F_1, F_2) : L \longrightarrow L'$  be a hom-Lie 2-algebra morphism, and  $S(L) = \mathcal{V}$ ,  $S(L') = \mathcal{V}'$ . Define  $S(F) = f = (f_0, f_1, f_2)$  as follows. Let  $f_0 = F_0$ ,  $f_1 = F_1|_{V_1 = \text{Ker}(s)}$  and define  $f_2$  by

$$f_2(x, y) = F_2(x, y) - i(s(F_2(x, y))).$$

It is not hard to deduce that  $f$  is a  $HL_\infty$ -algebra morphism. Furthermore,  $S$  also preserves the identity morphisms and the composition of morphisms. Thus,  $S$  is a functor from **HLie2** to **2HL** $_\infty$ .

We are left to show that there are natural isomorphisms  $\alpha : T \circ S \implies 1_{\mathbf{HLie2}}$  and  $\beta : S \circ T \implies 1_{\mathbf{2HL}_\infty}$ . For a hom-Lie 2-algebra  $(L, [\cdot, \cdot], J, \Phi)$ , applying the functor  $S$  to  $L$ , we obtain a 2-term  $HL_\infty$ -algebra  $\mathcal{V} = (V_1 = \text{Ker}(s) \xrightarrow{d=t|_{\text{Ker}(s)}} V_0 = L_0, l_2, l_3, \phi_0, \phi_1)$ . Applying the functor  $T$  to  $\mathcal{V}$ , we obtain a hom-Lie 2-algebra  $(L', [\cdot, \cdot]', \Phi', J')$ , with the space  $V_0$  of objects and the space  $V_0 \oplus \text{Ker}(s)$  of morphisms. Define  $\alpha_L : L' \longrightarrow L$  by setting

$$(\alpha_L)_0(x) = x, \quad (\alpha_L)_1(x, m) = i(x) + m.$$

It is obvious that  $\alpha_L$  is an isomorphism of 2-vector spaces. Furthermore, since  $[\cdot, \cdot]$  is a bilinear functor, we have  $[i(x), i(y)] = i([x, y])$ , and

$$[m, n] = [m \cdot_{\mathcal{V}} i(dm), i(0) \cdot_{\mathcal{V}} n] = [m, i(0)] \cdot_{\mathcal{V}} [i(dm), n] = [i(dm), n].$$

Therefore, we have

$$\begin{aligned} \alpha_L[(x, m), (y, n)]' &= \alpha_L(l_2(x, y), l_2(x, n) + l_2(m, y) + l_2(dm, n)) \\ &= \alpha_L([x, y], [i(x), n] + [m, i(y)] + [i(dm), n]) \\ &= i([x, y]) + [i(x), n] + [m, i(y)] + [i(dm), n] \\ &= [i(x), i(y)] + [i(x), n] + [m, i(y)] + [m, n] \\ &= [i(x) + m, i(y) + n] \\ &= [\alpha_L(x, m), \alpha_L(y, n)], \end{aligned}$$

which implies that  $\alpha_L$  is also a hom-Lie 2-algebra morphism with  $(\alpha_L)_2$  the identity isomorphism. Thus,  $\alpha_L$  is an isomorphism of hom-Lie 2-algebras. It is also easy to see that it is a natural isomorphism.

For a 2-term  $HL_{\infty}$ -algebra  $\mathcal{V} = (V_1 \xrightarrow{d} V_0, l_2, l_3, \phi_0, \phi_1)$ , applying the functor  $S$  to  $\mathcal{V}$ , we obtain a hom-Lie 2-algebra  $(L, [\cdot, \cdot], \Phi)$ . Applying the functor  $T$  to  $L$ , we obtain exactly the same 2-term  $HL_{\infty}$ -algebra  $\mathcal{V}$ . Thus,  $\beta_{\mathcal{V}} = \text{Id}_{\mathcal{V}} = (\text{Id}_{V_0}, \text{Id}_{V_1})$  is the natural isomorphism from  $T \circ S$  to  $1_{\mathbf{2HL}_{\infty}}$ . This finishes the proof. ■

## 4 Skeletal hom-Lie 2-algebras

Since we have proved that the category of hom-Lie 2-algebras and the category of 2-term  $HL_{\infty}$ -algebras are equivalent, in the following, when we say a hom-Lie 2-algebra, what we mean is a 2-term  $HL_{\infty}$ -algebra. In this section, first we give the classification of hom-Lie 2-algebras, and then we construct examples of skeletal hom-Lie 2-algebras, which are hom-analogues of string Lie 2-algebras, from quadratic hom-Lie algebras introduced in [4].

### 4.1 The classification of skeletal hom-Lie 2-algebras

A 2-term  $HL_{\infty}$ -algebra is called *skeletal* if  $d = 0$ . Let  $\mathcal{V}$  be a skeletal 2-term  $HL_{\infty}$ -algebra. By Condition (h) in Definition 3.5, we see that  $(V_0, l_2(\cdot, \cdot), \phi_0)$  is exactly a hom-Lie algebra. Define  $\rho_{\phi_1} : V_0 \longrightarrow \mathfrak{gl}(V_1)$  by

$$\rho_{\phi_1}(x)(m) = l_2(x, m), \quad \forall x \in V_0, m \in V_1. \quad (11)$$

**Proposition 4.1.** *Let  $\mathcal{V}$  be a skeletal 2-term  $HL_{\infty}$ -algebra, then the map  $\rho_{\phi_1}$  defined by (11) is a representation of the hom-Lie algebra  $(V_0, l_2(\cdot, \cdot), \phi_0)$  on  $V_1$  with respect to  $\phi_1$ .*

**Proof.** We only need to check that the two conditions in Definition 2.3 are satisfied. For any  $x \in V_0, m \in V_1$ , by Condition (g) in Definition 3.5, we have

$$l_2(\phi_0(x), \phi_1(m)) = \phi_1(l_2(x, m)),$$

which means that

$$\rho_{\phi_1}(\phi_0(x)) \circ \phi_1 = \phi_1 \circ \rho_{\phi_1}(x).$$

Thus Condition (i) in Definition 2.3 is satisfied. Furthermore, since  $\mathcal{V}$  is skeletal, by Condition (i) we have

$$l_2(\phi_0(x), l_2(y, m)) + l_2(\phi_0(y), l_2(m, x)) + l_2(\phi_1(m), l_2(x, y)) = 0,$$

which yields that

$$\rho_{\phi_1}(l_2(x, y)) \circ \phi_1 = \rho_{\phi_1}(\phi_0(x)) \circ \rho_{\phi_1}(y) - \rho_{\phi_1}(\phi_0(y)) \circ \rho_{\phi_1}(x).$$

Therefore, Condition (ii) in Definition 2.3 is satisfied. Thus  $\rho_{\phi_1}$  is a representation of the hom-Lie algebra  $(V_0, l_2(\cdot, \cdot), \phi_0)$  on  $V_1$  with respect to  $\phi_1$ . ■

**Theorem 4.2.** *There is a one-to-one correspondence between skeletal 2-term  $HL_\infty$ -algebras and druples  $((\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}}), W, A, \rho_A, \theta)$ , where  $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}})$  is a hom-Lie algebras,  $W$  is a vector space,  $A \in \mathfrak{gl}(W)$ ,  $\rho_A : \mathfrak{g} \rightarrow \mathfrak{gl}(W)$  is a representation of  $\mathfrak{g}$  on  $W$  with respect to  $A$ , and  $\theta$  is a 3-hom-cocycle of the hom-Lie algebra  $\mathfrak{g}$  with coefficients in the representation  $\rho_A$ .*

**Proof.** For any skeletal 2-term  $HL_\infty$ -algebra  $V$ ,  $(V_0, l_2(\cdot, \cdot), \phi_0)$  is a hom-Lie algebra. By Proposition 4.1,  $\rho_{\phi_1} : V_0 \rightarrow \mathfrak{gl}(V_1)$  defined by (11) is a representation of the hom-Lie algebra  $(V_0, l_2(\cdot, \cdot), \phi_0)$  on  $V_1$  with respect to  $\phi_1$ . Now we prove that  $l_3$  is a 3-hom-cocycle with respect to the representation  $\rho_{\phi_1}$  and thus any skeletal 2-term  $HL_\infty$ -algebra gives rise to a druple  $((V_0, l_2(\cdot, \cdot), \phi_0), V_1, \phi_1, \rho_{\phi_1}, l_3)$ . In fact, by Condition (j) in Definition 3.5, we have

$$\begin{aligned} & l_3(l_2(w, x), \phi_0(y), \phi_0(z)) + l_2(l_3(w, x, z), \phi_0^2(y)) \\ & + l_3(\phi_0(w), l_2(x, z), \phi_0(y)) + l_3(l_2(w, z), \phi_0(x), \phi_0(y)) \\ = & l_2(l_3(w, x, y), \phi_0^2(z)) + l_3(l_2(w, y), \phi_0(x), \phi_0(z)) + l_3(\phi_0(w), l_2(x, y), \phi_0(z)) \\ & + l_2(\phi_0^2(w), l_3(x, y, z)) + l_2(l_3(w, y, z), \phi_0^2(x)) + l_3(\phi_0(w), l_2(y, z), \phi_0(x)), \end{aligned}$$

which exactly means that

$$(d_{\rho_{\phi_1}} l_3)(w, x, y, z) = 0.$$

The converse part is easy to be checked and this finishes the proof. ■

## 4.2 The construction of skeletal hom-Lie 2-algebras from quadratic hom-Lie algebras

**Definition 4.3.** [4] *A quadratic hom-Lie algebra is hom-Lie algebra  $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}})$  together with a symmetric nondegenerate bilinear form  $B : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$ , such that for any  $x, y, z \in \mathfrak{g}$ , the following equalities are satisfied:*

$$B([x, y]_{\mathfrak{g}}, z) = -B([x, z]_{\mathfrak{g}}, y), \quad (12)$$

$$B(\phi_{\mathfrak{g}}(x), y) = B(x, \phi_{\mathfrak{g}}(y)). \quad (13)$$

Recall that a (quadratic) hom-Lie algebra is said to be involutive if  $\phi_{\mathfrak{g}}$  satisfies

$$\phi_{\mathfrak{g}}^2 = \text{Id}. \quad (14)$$

For a symmetric nondegenerate bilinear form  $B$ , there are close relations between conditions (13), (14), and

$$B(\phi_{\mathfrak{g}}(x), \phi_{\mathfrak{g}}(y)) = B(x, y). \quad (15)$$

**Lemma 4.4.** *Let  $B$  be a symmetric nondegenerate bilinear form on the hom-Lie algebra  $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}})$ . Consider the three conditions (13), (14) and (15), any two of them can imply the third one.*

**Proof.** If  $B$  satisfies (13) and (14), we have

$$B(\phi_{\mathfrak{g}}(x), \phi_{\mathfrak{g}}(y)) = B(x, \phi_{\mathfrak{g}}^2(y)) = B(x, y).$$

If  $B$  satisfies (13) and (15), on one hand, we have  $B(\phi_{\mathfrak{g}}(x), \phi_{\mathfrak{g}}(y)) = B(x, y)$ . On the other hand, we have  $B(\phi_{\mathfrak{g}}(x), \phi_{\mathfrak{g}}(y)) = B(x, \phi_{\mathfrak{g}}^2(y))$ . Thus, we have

$$B(x, (\phi_{\mathfrak{g}}^2 - \text{Id})(y)) = 0.$$

Since  $B$  is nondegenerate, we deduce that  $\phi_{\mathfrak{g}}^2 = \text{Id}$ .

If  $B$  satisfies (14) and (15), we have

$$B(\phi_{\mathfrak{g}}(x), y) = B(\phi_{\mathfrak{g}}(x), \phi_{\mathfrak{g}}^2 y) = B(x, \phi_{\mathfrak{g}}(y)).$$

This finishes the proof. ■

Let  $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}}, B)$  be an involutive quadratic hom-Lie algebra. Define  $l_3^B : \wedge^3 \mathfrak{g} \longrightarrow \mathbb{R}$  by

$$l_3^B(x, y, z) = B([x, y]_{\mathfrak{g}}, z). \quad (16)$$

By (12),  $l_3$  is skew-symmetric.

**Lemma 4.5.**  *$l_3^B$  is a 3-hom cocycle with coefficients in the trivial representation.*

**Proof.** First, by Lemma 4.4, we have

$$\begin{aligned} l_3^B(\phi_{\mathfrak{g}}(x), \phi_{\mathfrak{g}}(y), \phi_{\mathfrak{g}}(z)) &= B([\phi_{\mathfrak{g}}(x), \phi_{\mathfrak{g}}(y)]_{\mathfrak{g}}, \phi_{\mathfrak{g}}(z)) = B(\phi_{\mathfrak{g}}[x, y]_{\mathfrak{g}}, \phi_{\mathfrak{g}}(z)) \\ &= B([x, y]_{\mathfrak{g}}, z) = l_3^B(x, y, z), \end{aligned}$$

which implies that  $l_3$  is a 3-hom-cochain. Furthermore, by (12) and the hom-Jacobi identity, we have

$$\begin{aligned} &2d_T l_3^B(w, x, y, z) \\ &= 2(-l_3^B([w, x]_{\mathfrak{g}}, \phi_{\mathfrak{g}}(y), \phi_{\mathfrak{g}}(z)) + l_3^B([w, y]_{\mathfrak{g}}, \phi_{\mathfrak{g}}(x), \phi_{\mathfrak{g}}(z)) - l_3^B([w, z]_{\mathfrak{g}}, \phi_{\mathfrak{g}}(x), \phi_{\mathfrak{g}}(y)) \\ &\quad - l_3^B([x, y]_{\mathfrak{g}}, \phi_{\mathfrak{g}}(w), \phi_{\mathfrak{g}}(z)) + l_3^B([x, z]_{\mathfrak{g}}, \phi_{\mathfrak{g}}(w), \phi_{\mathfrak{g}}(y)) - l_3^B([y, z]_{\mathfrak{g}}, \phi_{\mathfrak{g}}(w), \phi_{\mathfrak{g}}(x))) \\ &= -B([w, x]_{\mathfrak{g}}, \phi_{\mathfrak{g}}(y)]_{\mathfrak{g}}, \phi_{\mathfrak{g}}(z)) + B([w, y]_{\mathfrak{g}}, \phi_{\mathfrak{g}}(x)]_{\mathfrak{g}}, \phi_{\mathfrak{g}}(z)) - B([w, z]_{\mathfrak{g}}, \phi_{\mathfrak{g}}(x)]_{\mathfrak{g}}, \phi_{\mathfrak{g}}(y)) \\ &\quad - B([x, y]_{\mathfrak{g}}, \phi_{\mathfrak{g}}(w)]_{\mathfrak{g}}, \phi_{\mathfrak{g}}(z)) + B([x, z]_{\mathfrak{g}}, \phi_{\mathfrak{g}}(w)]_{\mathfrak{g}}, \phi_{\mathfrak{g}}(y)) - B([y, z]_{\mathfrak{g}}, \phi_{\mathfrak{g}}(w)]_{\mathfrak{g}}, \phi_{\mathfrak{g}}(x)) \\ &\quad + B([w, x]_{\mathfrak{g}}, \phi_{\mathfrak{g}}(z)]_{\mathfrak{g}}, \phi_{\mathfrak{g}}(y)) - B([w, y]_{\mathfrak{g}}, \phi_{\mathfrak{g}}(z)]_{\mathfrak{g}}, \phi_{\mathfrak{g}}(x)) + B([w, z]_{\mathfrak{g}}, \phi_{\mathfrak{g}}(y)]_{\mathfrak{g}}, \phi_{\mathfrak{g}}(x)) \\ &\quad + B([x, y]_{\mathfrak{g}}, \phi_{\mathfrak{g}}(z)]_{\mathfrak{g}}, \phi_{\mathfrak{g}}(w)) - B([x, z]_{\mathfrak{g}}, \phi_{\mathfrak{g}}(y)]_{\mathfrak{g}}, \phi_{\mathfrak{g}}(w)) + B([y, z]_{\mathfrak{g}}, \phi_{\mathfrak{g}}(x)]_{\mathfrak{g}}, \phi_{\mathfrak{g}}(w)) \\ &= 0. \end{aligned}$$

Thus,  $l_3^B$  is a 3-hom-cocycle. ■

Now we are ready to construct an example of skeletal hom-Lie 2-algebras  $\mathcal{V} = (V_1 \xrightarrow{0} V_0, l_2, l_3, \phi_0, \phi_1)$  from an involutive quadratic hom-Lie algebra  $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}}, B)$  as follows. Let  $V_1 = \mathbb{R}$ ,  $V_0 = \mathfrak{g}$ ,  $\phi_0 = \phi_{\mathfrak{g}}$  and  $\phi_1 = \text{Id}$ . Define  $l_2$  by

$$l_2(x, y) = [x, y]_{\mathfrak{g}}, \quad l_2(x, m) = 0, \quad (17)$$

and define  $l_3$  by (16). By Lemma 4.5, it is straightforward to see that all the conditions in Definition 3.5 are satisfied. Therefore  $(\mathbb{R} \xrightarrow{0} \mathfrak{g}, l_2, l_3^B, \phi_{\mathfrak{g}}, \text{Id})$  is a skeletal hom-Lie 2-algebra for any involutive quadratic hom-Lie algebra  $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}}, B)$ .

In the following, we construct the hom-analogue of string Lie 2-algebras. We need some preparations. For any involutive hom-Lie algebra  $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}})$ ,  $(\mathfrak{g}_{\phi_{\mathfrak{g}}}, [\cdot, \cdot]_{\phi_{\mathfrak{g}}})$  is a Lie algebra [4, Theorem 5.1], where  $[\cdot, \cdot]_{\phi_{\mathfrak{g}}}$  is given by

$$[x, y]_{\phi_{\mathfrak{g}}} = [\phi_{\mathfrak{g}}(x), \phi_{\mathfrak{g}}(y)]_{\mathfrak{g}} = \phi_{\mathfrak{g}}([x, y]_{\mathfrak{g}}).$$

**Theorem 4.6.** *There is an inclusion from  $H^k(\mathfrak{g})$  to  $H^k(\mathfrak{g}_{\phi_{\mathfrak{g}}})$ , where  $H^k(\mathfrak{g})$  is the  $k$ -th cohomology group of the hom-Lie algebra  $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}})$  with the coefficients in the trivial representation, and  $H^k(\mathfrak{g}_{\phi_{\mathfrak{g}}})$  is the  $k$ -th cohomology group of the Lie algebra  $(\mathfrak{g}_{\phi_{\mathfrak{g}}}, [\cdot, \cdot]_{\phi_{\mathfrak{g}}})$  with the coefficients in the trivial representation.*

**Proof.** We only need to show that for any  $f \in Z_{\phi_{\mathfrak{g}}}^k(\mathfrak{g})$ , as a  $k$ -cochain of  $\mathfrak{g}_{\phi_{\mathfrak{g}}}$ ,  $f$  is also closed, and for any  $f \in B_{\phi_{\mathfrak{g}}}^k(\mathfrak{g})$ , as a  $k$ -cochain of  $\mathfrak{g}_{\phi_{\mathfrak{g}}}$ ,  $f$  is also exact. In fact, for any  $f \in Z_{\phi_{\mathfrak{g}}}^k(\mathfrak{g})$ , we have

$$\begin{aligned} f(\phi_{\mathfrak{g}}(x_1), \dots, \phi_{\mathfrak{g}}(x_k)) &= f(x_1, \dots, x_k), \\ d_T f(x_1, \dots, x_{k+1}) &= \sum_{i < j} (-1)^{i+j} f([x_i, x_j]_{\mathfrak{g}}, \phi_{\mathfrak{g}}(x_1), \dots, \widehat{x_i}, \dots, \widehat{x_j}, \dots, \phi_{\mathfrak{g}}(x_{k+1})) = 0. \end{aligned}$$

Since  $\phi_{\mathfrak{g}}^2 = \text{Id}$ , we have

$$\begin{aligned} 0 &= \sum_{i < j} (-1)^{i+j} f([\phi_{\mathfrak{g}}^2(x_i), \phi_{\mathfrak{g}}^2(x_j)]_{\mathfrak{g}}, \phi_{\mathfrak{g}}(x_1), \dots, \widehat{x_i}, \dots, \widehat{x_j}, \dots, \phi_{\mathfrak{g}}(x_{k+1})) \\ &= \sum_{i < j} (-1)^{i+j} f(\phi_{\mathfrak{g}}[\phi_{\mathfrak{g}}(x_i), \phi_{\mathfrak{g}}(x_j)]_{\mathfrak{g}}, \phi_{\mathfrak{g}}(x_1), \dots, \widehat{x_i}, \dots, \widehat{x_j}, \dots, \phi_{\mathfrak{g}}(x_{k+1})) \\ &= \sum_{i < j} (-1)^{i+j} f([\phi_{\mathfrak{g}}(x_i), \phi_{\mathfrak{g}}(x_j)]_{\mathfrak{g}}, x_1, \dots, \widehat{x_i}, \dots, \widehat{x_j}, \dots, x_{k+1}) \\ &= \sum_{i < j} (-1)^{i+j} f([x_i, x_j]_{\phi_{\mathfrak{g}}}, x_1, \dots, \widehat{x_i}, \dots, \widehat{x_j}, \dots, x_{k+1}) \\ &= d_{\mathfrak{g}_{\phi_{\mathfrak{g}}}} f(x_1, \dots, x_{k+1}), \end{aligned}$$

where  $d_{\mathfrak{g}_{\phi_{\mathfrak{g}}}}$  is the coboundary operator of the Lie algebra  $\mathfrak{g}_{\phi_{\mathfrak{g}}}$  with the coefficients in the trivial representation. Therefore, as a  $k$ -cochain of  $\mathfrak{g}_{\phi_{\mathfrak{g}}}$ ,  $f$  is also closed.

For any  $f \in B_{\phi_{\mathfrak{g}}}^k(\mathfrak{g})$ , assume that  $f = d_T h$ , for some  $h : \wedge^{k-1} \mathfrak{g} \rightarrow \mathbb{R}$  satisfying  $h \circ \phi_{\mathfrak{g}} = h$ . Similar as the above proof, we have

$$f(x_1, \dots, x_k) = d_T h(x_1, \dots, x_k) = d_{\mathfrak{g}_{\phi_{\mathfrak{g}}}} h(x_1, \dots, x_k),$$

which implies that, as a  $k$ -cochain of  $\mathfrak{g}_{\phi_{\mathfrak{g}}}$ ,  $f$  is also exact. This finishes the proof.  $\blacksquare$

Now let the involutive hom-Lie algebra  $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}})$  be semisimple<sup>2</sup>, then the Lie algebra  $(\mathfrak{g}_{\phi_{\mathfrak{g}}}, [\cdot, \cdot]_{\phi_{\mathfrak{g}}})$  is also semisimple [4]. Furthermore, the authors define a symmetric bilinear form  $B : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$  by

$$B(x, y) = \text{tr}(\text{ad}_x \circ \text{ad}_y), \quad (18)$$

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<sup>2</sup>For the notion of semisimple hom-Lie algebras, we refer to [4] and references therein

where  $\text{ad}_x$  is defined as usual:  $\text{ad}_x y = [x, y]_{\mathfrak{g}}$ , and  $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}}, B)$  is a semisimple quadratic involutive hom-Lie algebra. There is also the following relation

$$K_{\mathfrak{g}_{\phi_{\mathfrak{g}}}}(x, y) = B(\phi_{\mathfrak{g}}(x), y),$$

where  $K_{\mathfrak{g}_{\phi_{\mathfrak{g}}}}$  is the Killing form of the semisimple Lie algebra  $\mathfrak{g}_{\phi_{\mathfrak{g}}}$ .

**Corollary 4.7.** *Let  $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}})$  be a semisimple involutive hom-Lie algebra, then the cohomology class of  $l_3^B$  defined by (16) is not trivial, where  $B$  is given by (18).*

**Proof.** Since  $\phi_{\mathfrak{g}}^2 = \text{Id}$ , we have

$$K([x, y]_{\phi_{\mathfrak{g}}}, z) = B(\phi_{\mathfrak{g}}[x, y]_{\phi_{\mathfrak{g}}}, z) = B([\phi_{\mathfrak{g}}^2(x), \phi_{\mathfrak{g}}^2(y)]_{\mathfrak{g}}, z) = B([x, y]_{\mathfrak{g}}, z) = l_3^B(x, y, z).$$

Thus,  $l_3^B$  defined by (16), as a 3-cochain of the Lie algebra  $\mathfrak{g}_{\phi_{\mathfrak{g}}}$ , is exactly the Cartan 3-form of  $\mathfrak{g}_{\phi_{\mathfrak{g}}}$ . By Theorem 4.6, if  $l_3^B$  is exact, we deduce that the Cartan 3-form of the semisimple Lie algebra  $\mathfrak{g}_{\phi_{\mathfrak{g}}}$  is exact, this is a conflict. ■

**Definition 4.8.** *The hom-analogue of the string Lie 2-algebra associated to any semisimple involutive hom-Lie algebra  $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}})$  is the hom-Lie 2-algebra  $(\mathbb{R} \xrightarrow{0} \mathfrak{g}, l_2, l_3^B, \phi_{\mathfrak{g}}, \text{Id})$ , where  $l_2$ ,  $l_3^B$  and  $B$  are given by (17), (16) and (18) respectively.*

**Example 4.9.** *Consider the semisimple Lie algebra  $\mathfrak{sl}(2)$ , with basis  $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ ,  $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ , and  $C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$  satisfying the relation*

$$[A, B] = C, \quad [C, A] = 2A, \quad [B, C] = 2B.$$

*For any  $x \in \mathfrak{sl}(2)$ , let  $\phi(x) = -x^T$ , the minus of the transpose of  $x$ . Obviously,  $\phi$  is an involution map. Then  $(\mathfrak{sl}(2), [\cdot, \cdot]_{\phi}, \phi)$  is a semisimple involutive hom-Lie algebra. More precisely, we have*

$$[A, B]_{\phi} = [\phi(A), \phi(B)] = [-B, -A] = -C, \quad [C, A]_{\phi} = -2B, \quad [B, C]_{\phi} = -2A. \quad (19)$$

*It is easy to obtain that*

$$l_3^B(A, B, C) = B([A, B]_{\phi}, C) = -\text{tr}(\text{ad}_C^2) = 8. \quad (20)$$

*Therefore, we obtain a hom-analogue of the string Lie 2-algebra  $(\mathbb{R} \xrightarrow{0} \mathfrak{sl}(2), l_2, l_3^B, \phi, \text{Id})$ , where  $l_2$  and  $l_3^B$  are determined by (19) and (20).*

## 5 Strict hom-Lie 2-algebras

In this section, we introduce the notion of crossed modules of hom-Lie algebras, and we prove that there is a one-to-one correspondence between crossed modules of hom-Lie algebras and strict hom-Lie 2-algebras. Here what we mean a strict hom-Lie 2-algebra is a 2-term  $HL_{\infty}$ -algebra whose  $l_3$  is zero. Then we construct strict hom-Lie 2-algebras from hom-left-symmetric algebras. At last, we introduce the notion of symplectic hom-Lie algebras, and give the construction of strict hom-Lie 2-algebras from symplectic hom-Lie algebras.

## 5.1 Strict hom-Lie 2-algebras and crossed modules of hom-Lie algebras

**Definition 5.1.** A crossed module of hom-Lie algebras is a quadruple  $((\mathfrak{h}, [\cdot, \cdot]_{\mathfrak{h}}, \phi_{\mathfrak{h}}), (\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}}), dt, \varphi)$ , where  $(\mathfrak{h}, [\cdot, \cdot]_{\mathfrak{h}}, \phi_{\mathfrak{h}})$  and  $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}})$  are hom-Lie algebras,  $dt : \mathfrak{h} \rightarrow \mathfrak{g}$  is a hom-Lie algebra morphism and  $\varphi$  is a representation of the hom-Lie algebra  $\mathfrak{g}$  on  $\mathfrak{h}$ , such that

$$dt(\varphi_x(m)) = [x, dt(m)]_{\mathfrak{g}}, \quad (21)$$

$$\varphi_{dt(m)}(m') = [m, m']_{\mathfrak{h}}. \quad (22)$$

**Lemma 5.2.** Let  $((\mathfrak{h}, \phi_{\mathfrak{h}}), (\mathfrak{g}, \phi_{\mathfrak{g}}), dt, \varphi)$  be a crossed module of hom-Lie algebras, then we have

$$\varphi_{\phi_{\mathfrak{g}}(x)}([m, n]_{\mathfrak{h}}) = [\varphi_x m, \phi_{\mathfrak{h}}(n)]_{\mathfrak{h}} + [\phi_{\mathfrak{h}}(m), \varphi_x n]_{\mathfrak{h}}. \quad (23)$$

**Proof.** By the fact that  $\varphi$  is a representation, we have

$$\varphi_{[x, y]_{\mathfrak{g}}} \circ \phi_{\mathfrak{h}} = \varphi_{\phi_{\mathfrak{g}}(x)} \circ \varphi_y - \varphi_{\phi_{\mathfrak{g}}(y)} \circ \varphi_x.$$

Let  $y = dt(m)$ , by (21) and (22), we obtain

$$\varphi_{dt(\varphi_x m)} \phi_{\mathfrak{h}}(n) = \varphi_{\phi_{\mathfrak{g}}(x)} \circ \varphi_{dt(m)} n - \varphi_{\phi_{\mathfrak{g}}(dt(m))} \circ \varphi_x n,$$

which implies that

$$[\varphi_x m, \phi_{\mathfrak{h}}(n)]_{\mathfrak{h}} = \varphi_{\phi_{\mathfrak{g}}(x)}[m, n]_{\mathfrak{h}} - [\phi_{\mathfrak{h}}(m), \varphi_x n]_{\mathfrak{h}}. \blacksquare$$

**Remark 5.3.** If  $\phi_{\mathfrak{g}} = \text{Id}$  and  $\phi_{\mathfrak{h}} = \text{Id}$ , i.e.  $(\mathfrak{h}, \mathfrak{g}, dt, \varphi)$  is a crossed module of Lie algebras, we deduce that  $\varphi$  must act as a derivation by the above proof.

**Theorem 5.4.** There is a one-to-one correspondence between strict hom-Lie 2-algebras and crossed modules of hom-Lie algebras.

**Proof.** Let  $(V_1 \xrightarrow{d} V_0, l_2, l_3 = 0, \phi_0, \phi_1)$  be a strict hom-Lie 2-algebra, we construct a crossed module of hom-Lie algebra as follows. Let  $\mathfrak{g} = V_0$  with the bracket operation  $[\cdot, \cdot]_{\mathfrak{g}} = l_2 : V_0 \times V_0 \rightarrow V_0$ , and linear transformation  $\phi_{\mathfrak{g}} = \phi_0$ . Let  $\mathfrak{h} = V_1$  with the bracket operation  $[\cdot, \cdot]_{\mathfrak{h}} : V_1 \times V_1 \rightarrow V_1$  given by

$$[m, n]_{\mathfrak{h}} = l_2(dm, n),$$

and linear transformation  $\phi_{\mathfrak{h}} = \phi_1$ . Furthermore, let  $dt = d$ .

By (a), (f) and (h), it is obvious that  $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \phi_0)$  is a hom-Lie algebra. By (b) and (e), the bracket operation  $[\cdot, \cdot]_{\mathfrak{h}}$  is well defined. By (g), we have

$$\phi_1([m, n]_{\mathfrak{h}}) = \phi_1(l_2(dm, n)) = l_2(\phi_0(dm), \phi_1(n)) = l_2(d \circ \phi_1(m), \phi_1(n)) = [\phi_1(m), \phi_1(n)]_{\mathfrak{h}},$$

which implies that  $\phi_1$  is an algebra morphism with respect to  $[\cdot, \cdot]_{\mathfrak{h}}$ . By (i), we have

$$\begin{aligned} & [\phi_1(m), [n, p]_{\mathfrak{h}}]_{\mathfrak{h}} + [\phi_1(n), [p, m]_{\mathfrak{h}}]_{\mathfrak{h}} + [\phi_1(p), [m, n]_{\mathfrak{h}}]_{\mathfrak{h}} \\ &= l_2(d \circ \phi_1(m), l_2(dn, p)) + l_2(d \circ \phi_1(n), l_2(dp, m)) + l_2(d \circ \phi_1(p), l_2(dm, n)) \\ &= l_2(\phi_0(dm), l_2(dn, p)) + l_2(\phi_0(dn), l_2(dp, m)) + l_2(\phi_0(dp), l_2(dm, n)) \\ &= l_2(\phi_0(dm), l_2(dn, p)) + l_2(\phi_0(dn), l_2(p, dm)) + l_2(\phi_1(p), l_2(dm, dn)) \\ &= 0. \end{aligned}$$



Thus,  $(\mathfrak{h}, [\cdot, \cdot]_{\mathfrak{h}}, \phi_1)$  is a hom-Lie algebra. By (d), it is obvious that  $dt$  is a morphism of hom-Lie algebras. At last, define  $\varphi : \mathfrak{g} \times \mathfrak{h} \longrightarrow \mathfrak{h}$  by

$$\varphi_x m = l_2(x, m).$$

By (g), we have  $\varphi_{\phi_{\mathfrak{g}}(x)} \phi_{\mathfrak{h}}(m) = \phi_{\mathfrak{h}} \varphi_x m$ . By (i), we have

$$\begin{aligned} & \varphi_{[x, y]_{\mathfrak{g}}} \phi_{\mathfrak{h}}(m) - \varphi_{\phi_{\mathfrak{g}}(x)} \circ \varphi_y m + \varphi_{\phi_{\mathfrak{g}}(y)} \circ \varphi_x m \\ &= l_2(l_2(x, y), \phi_1(m)) - l_2(\phi_0(x), l_2(y, m)) + l_2(\phi_0(y), l_2(x, m)) = 0. \end{aligned}$$

Thus,  $\varphi$  is a representation. By (d), we see that the equality (21) holds. By the definition of  $\varphi$  and  $[\cdot, \cdot]_{\mathfrak{h}}$ , it is obvious that the equality (22) holds. Therefore,  $((\mathfrak{h}, [\cdot, \cdot]_{\mathfrak{h}}, \phi_{\mathfrak{h}}), (\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}}), dt, \varphi)$  is a crossed module of hom-Lie algebras.

Conversely, given a crossed module of hom-Lie algebras  $((\mathfrak{h}, [\cdot, \cdot]_{\mathfrak{h}}, \phi_{\mathfrak{h}}), (\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}}), dt, \varphi)$ , we obtain a strict hom-Lie 2-algebra as follows. Let  $V_0 = \mathfrak{g}$ ,  $\phi_0 = \phi_{\mathfrak{g}}$ ,  $V_1 = \mathfrak{h}$ ,  $\phi_1 = \phi_{\mathfrak{h}}$  and  $d = dt$ . Define  $l_2 : V_i \times V_j \longrightarrow V_{i+j}$  by

$$l_2(x, y) = [x, y]_{\mathfrak{g}}, \quad l_2(x, m) = -l_2(m, x) = \varphi_x m, \quad l_2(m, n) = 0.$$

The crossed module structure gives various conditions of strict hom-Lie 2-algebras. We omit the details. ■

First we have the following trivial example of strict Lie 2-algebras.

**Example 5.5.** For any hom-Lie algebra  $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}})$ ,  $(\mathfrak{g} \xrightarrow{0} \mathfrak{g}, l_2 = [\cdot, \cdot]_{\mathfrak{g}}, \phi_0 = \phi_{\mathfrak{g}}, \phi_1 = \phi_{\mathfrak{g}})$  is a strict hom-Lie 2-algebra.

## 5.2 The construction of strict hom-Lie 2-algebras from hom-left-symmetric algebras

Hom-left-symmetric algebras, or hom-pre-Lie algebras were first introduced in [9], and then further studied in [16] and [17].

**Definition 5.6.** A hom-left-symmetric algebra is a triple  $(V, \star, \phi)$ , where  $V$  is a vector space,  $\star : V \times V \longrightarrow V$  is a bilinear map, and  $\phi \in \mathfrak{gl}(V)$  such that the following equalities are satisfied:

$$\phi(x \star y) = \phi(x) \star \phi(y), \quad (24)$$

$$\phi(x) \star (y \star z) - (x \star y) \star \phi(z) = \phi(y) \star (x \star z) - (y \star x) \star \phi(z). \quad (25)$$

Let  $(V, \star, \phi)$  be a hom-left-symmetric algebra, define  $[\cdot, \cdot]_V : V \wedge V \longrightarrow V$  by

$$[x, y]_V = x \star y - y \star x, \quad (26)$$

and define  $\rho_{\phi} : V \longrightarrow \mathfrak{gl}(V)$  by

$$\rho_{\phi}(x)(y) = x \star y. \quad (27)$$

**Proposition 5.7.** With the above notations,  $(V, [\cdot, \cdot]_V, \phi)$  is a hom-Lie algebra, which is called the sub-adjacent hom-Lie algebra of the hom-left-symmetric algebra  $(V, \star, \phi)$ , and  $\rho_{\phi}$  is a representation of the hom-Lie algebra  $V$  on the vector space  $V$  with respect to  $\phi$ . Moreover, if  $\phi^2 = \text{Id}$ , i.e.  $(V, [\cdot, \cdot]_V, \phi)$  is an involutive hom-Lie algebra, then the map  $\rho_{\phi}^* : V \longrightarrow \mathfrak{gl}(V^*)$  defined by

$$\langle \rho_{\phi}^*(x)(\xi), y \rangle = -\langle \xi, \rho_{\phi}(x)(y) \rangle,$$

is also a representation of  $(V, [\cdot, \cdot]_V, \phi)$  on the vector space  $V^*$  with respect to  $\phi^*$ .

**Proof.** The first part follows from straightforward computations. As for the second part, first we should note that, in general, for a representation  $\rho_A$  of the hom-Lie algebra  $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}})$  on the vector space  $U$  with respect to  $A \in \mathfrak{gl}(U)$ , the induced map  $\rho_A^* : \mathfrak{g} \rightarrow \mathfrak{gl}(U^*)$ ,  $\langle \rho_A^*(x)\xi, u \rangle = -\langle \xi, \rho_A(x)(u) \rangle$ , for any  $\xi \in U^*$  and  $u \in U$ , is a representation iff ([4, Proposition 2.5])

$$A \circ \rho_A([x, y]_{\mathfrak{g}}) = \rho_A(x) \circ \rho_A(\phi_{\mathfrak{g}}(y)) - \rho_A(y) \circ \rho_A(\phi_{\mathfrak{g}}(x)).$$

Consider the sub-adjacent hom-Lie algebra  $(V, [\cdot, \cdot]_V, \phi)$ , if  $\phi^2 = \text{Id}$ , the above condition reduces to

$$\begin{aligned} (\phi(x) \star \phi(y)) \star \phi(z) - (\phi(y) \star \phi(x)) \star \phi(z) &= x \star (\phi(y) \star z) - y \star (\phi(x) \star z) \\ &= \phi^2(x) \star (\phi(y) \star z) - \phi^2(y) \star (\phi(x) \star z), \end{aligned}$$

which holds naturally by (25). This finishes the proof. ■

The following procedure provides a way to construct examples of strict hom-Lie 2-algebras from hom-left-symmetric algebras.

**Proposition 5.8.** *Let  $(V, \star, \phi)$  be a hom-left-symmetric algebra, for any linear map  $d : V \rightarrow V$  satisfying*

$$d \circ \phi = \phi \circ d, \quad (28)$$

$$dx \star y = x \star dy, \quad (29)$$

$$d(x \star y) = x \star dy - dy \star x, \quad (30)$$

define  $l_2$  on the 2-term complex of vector spaces  $V_1 = V \xrightarrow{d} V_0 = V$  by

$$\begin{cases} l_2(x, y) &= [x, y]_V, & \forall x, y \in V_0 \\ l_2(x, y) &= -l_2(y, x) = x \star y, & \forall x \in V_0, y \in V_1 \\ l_2(x, y) &= 0, & \forall x, y \in V_1. \end{cases} \quad (31)$$

Then  $(V_1 = V \xrightarrow{d} V_0 = V, l_2, \phi_0 = \phi, \phi_1 = \phi)$  is a strict hom-Lie 2-algebra.

**Proof.** By the definition of  $l_2$ , conditions (a), (b) and (c) in Definition 3.5 are satisfied obviously. By Proposition 5.7, conditions (f), (g), (h) and (i) are also satisfied. At last, (29) and (30) imply that conditions (d) and (e) hold. ■

### 5.3 The construction of strict hom-Lie 2-algebras from symplectic hom-Lie algebras

**Definition 5.9.** *Let  $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}})$  be a regular hom-Lie algebra,  $\omega \in \wedge^2 \mathfrak{g}^*$  is called a symplectic structure on  $\mathfrak{g}$  if*

- $\omega$  is nondegenerate, i.e. the induced skewsymmetric map  $\omega^\sharp : \mathfrak{g} \rightarrow \mathfrak{g}^*$ ,  $\langle \omega^\sharp(x), y \rangle = \omega(x, y)$ , is nondegenerate;
- $\omega$  is a 2-hom-cocycle, i.e. we have  $\omega \circ \phi_{\mathfrak{g}} = \omega$ , and  $d_T \omega = 0$ :

$$\omega(\phi_{\mathfrak{g}}(x), [y, z]_{\mathfrak{g}}) + \omega(\phi_{\mathfrak{g}}(y), [z, x]_{\mathfrak{g}}) + \omega(\phi_{\mathfrak{g}}(z), [x, y]_{\mathfrak{g}}) = 0. \quad (32)$$

$(\mathfrak{g}, \omega)$  is called a symplectic hom-Lie algebra if  $\omega$  is a symplectic structure on  $\mathfrak{g}$ .

Define a bilinear map  $\star : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$  on the regular symplectic hom-Lie algebra  $(\mathfrak{g}, \omega)$  by

$$\omega(x \star y, \phi_{\mathfrak{g}}(z)) = -\omega(\phi_{\mathfrak{g}}(y), [x, z]_{\mathfrak{g}}). \quad (33)$$

By the fact  $\omega$  is closed, we have

$$\omega(x \star y - y \star x, \phi_{\mathfrak{g}}(z)) = -\omega(\phi_{\mathfrak{g}}(y), [x, z]_{\mathfrak{g}}) + \omega(\phi_{\mathfrak{g}}(x), [y, z]_{\mathfrak{g}}) = \omega([x, y]_{\mathfrak{g}}, \phi_{\mathfrak{g}}(z)),$$

which implies that

$$[x, y]_{\mathfrak{g}} = x \star y - y \star x, \quad (34)$$

since  $\phi_{\mathfrak{g}}$  is nondegenerate.

**Proposition 5.10.** *Let  $((\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}}), \omega)$  be a regular symplectic hom-Lie algebra, then  $(\mathfrak{g}, \star, \phi_{\mathfrak{g}})$  is a hom-left-symmetric algebra. Furthermore, the hom-Lie algebra  $\mathfrak{g}$  is its sub-adjacent hom-Lie algebra.*

**Proof.** By (33), we have

$$\begin{aligned} \omega(\phi_{\mathfrak{g}}(x) \star \phi_{\mathfrak{g}}(y), \phi_{\mathfrak{g}}^2(z)) &= -\omega(\phi_{\mathfrak{g}}^2(y), [\phi_{\mathfrak{g}}(x), \phi_{\mathfrak{g}}(z)]_{\mathfrak{g}}) = -\omega(\phi_{\mathfrak{g}}(y), [x, z]_{\mathfrak{g}}) \\ &= \omega(x \star y, \phi_{\mathfrak{g}}(z)) = \omega(\phi_{\mathfrak{g}}(x \star y), \phi_{\mathfrak{g}}^2(z)). \end{aligned}$$

Since both  $\omega$  and  $\phi_{\mathfrak{g}}$  are nondegenerate, we deduce that  $\phi_{\mathfrak{g}}(x \star y) = \phi_{\mathfrak{g}}(x) \star \phi_{\mathfrak{g}}(y)$ . We have

$$\begin{aligned} &\omega(\phi_{\mathfrak{g}}(x) \star (y \star z) - (x \star y) \star \phi_{\mathfrak{g}}(z) - \phi_{\mathfrak{g}}(y) \star (x \star z) + (y \star x) \star \phi_{\mathfrak{g}}(z), \phi_{\mathfrak{g}}^2(w)) \\ &= -\omega(\phi_{\mathfrak{g}}(y \star z), [\phi_{\mathfrak{g}}(x), \phi_{\mathfrak{g}}(w)]_{\mathfrak{g}}) + \omega(\phi_{\mathfrak{g}}^2(z), [x \star y, \phi_{\mathfrak{g}}(w)]_{\mathfrak{g}}) \\ &\quad + \omega(\phi_{\mathfrak{g}}(x \star z), [\phi_{\mathfrak{g}}(y), \phi_{\mathfrak{g}}(w)]_{\mathfrak{g}}) - \omega(\phi_{\mathfrak{g}}^2(z), [y \star x, \phi_{\mathfrak{g}}(w)]_{\mathfrak{g}}) \\ &= -\omega(\phi_{\mathfrak{g}}(y) \star \phi_{\mathfrak{g}}(z), \phi_{\mathfrak{g}}[x, w]_{\mathfrak{g}}) + \omega(\phi_{\mathfrak{g}}^2(z), [[x, y]_{\mathfrak{g}}, \phi_{\mathfrak{g}}(w)]_{\mathfrak{g}}) + \omega(\phi_{\mathfrak{g}}(x) \star \phi_{\mathfrak{g}}(z), \phi_{\mathfrak{g}}[y, w]_{\mathfrak{g}}) \\ &= \omega(\phi_{\mathfrak{g}}^2(z), [\phi_{\mathfrak{g}}(y), [x, w]_{\mathfrak{g}}]_{\mathfrak{g}}) + \omega(\phi_{\mathfrak{g}}^2(z), [[x, y]_{\mathfrak{g}}, \phi_{\mathfrak{g}}(w)]_{\mathfrak{g}}) - \omega(\phi_{\mathfrak{g}}^2(z), [\phi_{\mathfrak{g}}(x), [y, w]_{\mathfrak{g}}]_{\mathfrak{g}}) \\ &= 0, \end{aligned}$$

which implies that

$$\phi_{\mathfrak{g}}(x) \star (y \star z) - (x \star y) \star \phi_{\mathfrak{g}}(z) - \phi_{\mathfrak{g}}(y) \star (x \star z) + (y \star x) \star \phi_{\mathfrak{g}}(z) = 0.$$

Thus,  $(\mathfrak{g}, \star, \phi_{\mathfrak{g}})$  is a hom-left-symmetric algebra. By (34), the second conclusion is obvious. ■

The following theorem provides a procedure to construct strict hom-Lie 2-algebras from involutive symplectic hom-Lie algebras.

**Theorem 5.11.** *Let  $((\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}}), \omega)$  be a involutive symplectic hom-Lie algebra, and  $(\mathfrak{g}, \star, \phi_{\mathfrak{g}})$  be the induced hom-left-symmetric algebra as in Proposition 5.10. On the complex of vector spaces  $\mathfrak{g}^* \xrightarrow{\phi_{\mathfrak{g}} \circ (\omega^{\sharp})^{-1}} \mathfrak{g}$ , define  $l_2$  by*

$$\begin{cases} l_2(x, y) &= [x, y]_{\mathfrak{g}}, & \forall x, y \in \mathfrak{g} \\ l_2(x, \xi) &= -l_2(\xi, x) = \rho_{\phi_{\mathfrak{g}}}^*(x)\xi, & \forall x \in \mathfrak{g}, \xi \in \mathfrak{g}^* \\ l_2(\xi, \eta) &= 0, & \forall \xi, \eta \in \mathfrak{g}^*, \end{cases} \quad (35)$$

where  $\rho_{\phi_{\mathfrak{g}}}^*$  is the dual representation of  $\rho_{\phi_{\mathfrak{g}}}$  given by (27), then  $(\mathfrak{g}^* \xrightarrow{\phi_{\mathfrak{g}} \circ (\omega^{\sharp})^{-1}} \mathfrak{g}, l_2, \phi_0 = \phi_{\mathfrak{g}}, \phi_1 = \phi_{\mathfrak{g}}^*)$  is a strict hom-Lie 2-algebra.

**Proof.** Similar as the proof of Proposition 5.8, we only need to show that

$$\phi_{\mathfrak{g}} \circ (\omega^{\sharp})^{-1} \circ \phi_{\mathfrak{g}}^* = \phi_{\mathfrak{g}}^2 \circ (\omega^{\sharp})^{-1}, \quad (36)$$

$$\phi_{\mathfrak{g}} \circ (\omega^{\sharp})^{-1} l_2(x, \xi) = l_2(x, \phi_{\mathfrak{g}} \circ (\omega^{\sharp})^{-1}(\xi)), \quad (37)$$

$$l_2(\phi_{\mathfrak{g}} \circ (\omega^{\sharp})^{-1}(\xi), \eta) = l_2(\xi, \phi_{\mathfrak{g}} \circ (\omega^{\sharp})^{-1}(\eta)). \quad (38)$$

The equality (36) is equivalent to

$$\langle (\omega^{\sharp})^{-1} \circ \phi_{\mathfrak{g}}^* \xi, \eta \rangle = \langle \phi_{\mathfrak{g}} \circ (\omega^{\sharp})^{-1} \xi, \eta \rangle.$$

Let  $\xi = \omega^{\sharp}(x)$  and  $\eta = \omega^{\sharp}(y)$ , since  $\omega$  is skew-symmetric, we have

$$\begin{aligned} \langle (\omega^{\sharp})^{-1} \circ \phi_{\mathfrak{g}}^* \xi, \eta \rangle &= -\langle \phi_{\mathfrak{g}}^*(\omega^{\sharp}(x)), y \rangle = -\langle \omega^{\sharp}(x), \phi_{\mathfrak{g}}(y) \rangle = \omega(\phi_{\mathfrak{g}}(y), x), \\ \langle \phi_{\mathfrak{g}} \circ (\omega^{\sharp})^{-1} \xi, \eta \rangle &= \langle \phi_{\mathfrak{g}}(x), \omega^{\sharp}(y) \rangle = \omega(y, \phi_{\mathfrak{g}}(x)). \end{aligned}$$

Since  $\omega \circ \phi_{\mathfrak{g}} = \omega$  and  $\phi_{\mathfrak{g}}^2 = \text{Id}$ , by Lemma 4.4, we deduce that  $\omega(y, \phi_{\mathfrak{g}}(x)) = \omega(\phi_{\mathfrak{g}}(y), x)$ . Therefore, (36) holds.

The equality (37) is equivalent to

$$\langle (\omega^{\sharp})^{-1} \circ \phi_{\mathfrak{g}}^* l_2(x, \xi), \eta \rangle = \langle l_2(x, \phi_{\mathfrak{g}} \circ (\omega^{\sharp})^{-1}(\xi)), \eta \rangle.$$

Let  $\xi = \omega^{\sharp}(y)$  and  $\eta = \omega^{\sharp}(z)$ , we have

$$\begin{aligned} \langle (\omega^{\sharp})^{-1} \circ \phi_{\mathfrak{g}}^* l_2(x, \xi), \eta \rangle &= -\langle \phi_{\mathfrak{g}}^* \rho_{\phi_{\mathfrak{g}}}^*(x)(\omega^{\sharp}(y)), z \rangle = \langle \omega^{\sharp}(y), \rho_{\phi_{\mathfrak{g}}}(x)(\phi_{\mathfrak{g}}(z)) \rangle = \omega(y, x \star \phi_{\mathfrak{g}}(z)), \\ \langle l_2(x, \phi_{\mathfrak{g}} \circ (\omega^{\sharp})^{-1}(\xi)), \eta \rangle &= \langle [x, \phi_{\mathfrak{g}}(y)]_{\mathfrak{g}}, \omega^{\sharp}(z) \rangle = \omega(z, [x, \phi_{\mathfrak{g}}(y)]_{\mathfrak{g}}). \end{aligned}$$

Since  $\phi_{\mathfrak{g}}^2 = \text{Id}$ , we have

$$\omega(y, x \star \phi_{\mathfrak{g}}(z)) = -\omega(\phi_{\mathfrak{g}}^2(x) \star \phi_{\mathfrak{g}}(z), \phi_{\mathfrak{g}}^2(y)) = \omega(\phi_{\mathfrak{g}}^2(z), [\phi_{\mathfrak{g}}^2(x), \phi_{\mathfrak{g}}(y)]_{\mathfrak{g}}) = \omega(z, [x, \phi_{\mathfrak{g}}(y)]_{\mathfrak{g}}),$$

which implies that (37) holds.

At last, let  $\xi = \omega^{\sharp}(x)$  and  $\eta = \omega^{\sharp}(y)$ , the equality (38) is equivalent to

$$\langle l_2(\phi_{\mathfrak{g}}(x), \omega^{\sharp}(y)), z \rangle = \langle l_2(\omega^{\sharp}(x), \phi_{\mathfrak{g}}(y)), z \rangle.$$

We have

$$\begin{aligned} \langle l_2(\phi_{\mathfrak{g}}(x), \omega^{\sharp}(y)), z \rangle &= -\langle \omega^{\sharp}(y), \phi_{\mathfrak{g}}(x) \star z \rangle = \omega(\phi_{\mathfrak{g}}(x) \star z, y) = -\omega(\phi_{\mathfrak{g}}(z), [\phi_{\mathfrak{g}}(x), \phi_{\mathfrak{g}}(y)]_{\mathfrak{g}}), \\ \langle l_2(\omega^{\sharp}(x), \phi_{\mathfrak{g}}(y)), z \rangle &= \langle \omega^{\sharp}(x), \phi_{\mathfrak{g}}(y) \star z \rangle = \omega(x, \phi_{\mathfrak{g}}(y) \star z) = \omega(\phi_{\mathfrak{g}}(z), [\phi_{\mathfrak{g}}(y), \phi_{\mathfrak{g}}(x)]_{\mathfrak{g}}). \end{aligned}$$

Thus, (38) holds. The proof is completed. ■

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